Hawaii State Math Bowl XXXVI

Brigham Young University – Hawaii

Laie, HI

May 3, 2014
Stop!

Do not continue until directed to do so!
Round #1
(10 minutes)
Problem A (4 points)

The Body Mass Index (BMI) varies directly with a person’s weight (pounds) and inversely with the square of the person’s height (inches). Given that a 6-foot tall man weighing 180 pounds has a BMI of 24.0, what is the BMI of a woman weighing 120 pounds with a height of 5 feet 4 inches?

Problem B (5 points)

Find the remaining real zero for the 4th degree polynomial function with the following properties: it only has real coefficients, 1/2 and 1 + 2i are two of its zeros, its y-intercept is −30, and its leading coefficient is 2.

Problem C (6 points)

What is the remainder when \( x^{2014} + x^{2013} \) is divided by \( x^2 + 1 \)?
Round #1
(10 minutes)

Answer: Problem A (4 points)

Answer: Problem B (5 points)

Answer: Problem C (6 points)
Answer: Problem A (4 points)
20.25

Answer: Problem B (5 points)
−6

Answer: Problem C (6 points)
x − 1
Stop!
Do not continue until directed to do so!
Round #2

(5 points/5 minutes)
Using the following figure (not drawn to scale), determine the value of \( m\angle A + m\angle B + m\angle C + m\angle D \).
Using the following figure (not drawn to scale), determine the value of $m\angle A + m\angle B + m\angle C + m\angle D$.

Answer: 280°
Using the following figure (not drawn to scale), determine the value of \( m\angle A + m\angle B + m\angle C + m\angle D \).

Answer: \( 280^\circ \)
Solution:
It can be seen directly that

\[ A + D + 125° = 180° \implies A + D = 55°. \]

Next, \( B + C + E + F = 360° \) and \( E + F + 45° = 180° \). Thus, \( B + C = 225° \). It follows that

\[ A + B + C + D = 225° + 55° = 280° \]
Stop!

Do not continue until directed to do so!
Round #3

(6 points/6 minutes)
The sum of the lengths of the twelve edges of a closed rectangular box is 140, and the distance from one corner of the box to the farthest corner is 21. The total surface area of the box is 784.
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**Answer:** 784
The sum of the lengths of the twelve edges of a closed rectangular box is 140, and the distance from one corner of the box to the farthest corner is 21. The total surface area of the box is

Answer:

784
Solution:

Let $h, w,$ and $l$ be the height, width, and length of the box. Because the sum of the lengths of all twelve edges is 140, then $4h + 4w + 4l = 140$, and therefore $h + w + l = 35$. Because the distance from one corner to the most distant one is 21, then $h^2 + w^2 + l^2 = 21^2$. The surface area, $S$, of the box equals $2lh + 2lw + 2hw$. Now,

$$ (h + w + l)^2 = h^2 + w^2 + l^2 + 2lh + 2lw + 2hw $$

$$ 35^2 = 21^2 + S $$

$$ 784 = S $$
Stop!
Do not continue until directed to do so!
Round #4

(7 points/7 minutes)
If \(3 \sin \theta + 4 \cos \theta = 5\), then \(\tan \theta = \)?
Round #4
(7 points/7 minutes)

If $3 \sin \theta + 4 \cos \theta = 5$, then $\tan \theta = \frac{3}{4}$.

Answer:
If $3 \sin \theta + 4 \cos \theta = 5$, then $\tan \theta =$?

**Answer:** \[
\frac{3}{4}
\]
Solution:

\[ 3 \sin \theta + 4 \cos \theta = 5 \]

\[ 4 \cos \theta = 5 - 3 \sin \theta \]

\[ 16 \cos^2 \theta = 25 - 30 \sin \theta + 9 \sin^2 \theta \]

\[ 16(1 - \sin^2 \theta) = 25 - 30 \sin \theta + 9 \sin^2 \theta \]

\[ 0 = 25 \sin^2 \theta - 30 \sin \theta + 9 \]

Using the quadratic formula gives

\[ \sin \theta = \frac{30 \pm \sqrt{30^2 - 4 \cdot 9 \cdot 25}}{2(25)} = \frac{30}{50} = \frac{3}{5} \]
It must be that \( \cos \theta > 0 \) (otherwise the sum of 5 is impossible). This gives

\[
\cos \theta = \sqrt{1 - \left( \frac{3}{5} \right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}
\]

It follows that \( \tan \theta = \frac{3}{4} \).
Stop!

Do not continue until directed to do so!
Round #5
(8 minutes)
Problem A (Algebra-Plane Geometry/6 points)
In the figure, $ABCD$ is a square with side length of $a$. $E$ is the midpoint of $BC$. $AE$ intersects the diagonal $BD$ at $F$. Find the area of triangle $ABF$.

Problem B (Trigonometry-Analytical Geometry/6 points)
Find the sum of all $x$-values satisfying the equation $2 \cos 3x + 1 = 0$ on the interval $[0, \pi]$.

Problem C (Probability-Statistics/6 points)
Urn A contains 4 white balls and 6 black balls. Urn B contains 3 white balls and 5 black balls. A ball (the transferred ball) is drawn from urn A and then transferred to urn B. A ball (the second ball) is then drawn from urn B. What is the probability that the transferred ball was black given that the second ball drawn was white?
Answer: Problem A (6 points)

Answer: Problem B (6 points)

Answer: Problem C (6 points)
Answer: Problem A (6 points)
\[
\frac{a^2}{6}
\]

Answer: Problem B (6 points)
\[
\frac{14\pi}{9}
\]

Answer: Problem C (6 points)
\[
\frac{9}{17}
\]
Stop!

Do not continue until directed to do so!
Round #6

(9 points/9 minutes)
If \( r \) and \( m \) are positive integers, \( r \) is a quadratic residue of \( m \) if there exists an integer \( x \) such that the remainder when \( x^2 \) is divided by \( m \) is equal to the remainder when \( r \) is divided by \( m \). For example, \( r = 86 \) is a quadratic residue of \( m = 7 \) because if we let \( x = 4 \) then \( x^2 = 16 \), and 16 and 86 each have a remainder of 2 when divided by 7. This is also written as there exists an integer \( x \) such that \( x^2 \equiv r (\text{mod } m) \). Find the sum of the smallest three positive prime numbers that are quadratic residues of \( m = 5 \).
If \( r \) and \( m \) are positive integers, \( r \) is a quadratic residue of \( m \) if there exists an integer \( x \) such that the remainder when \( x^2 \) is divided by \( m \) is equal to the remainder when \( r \) is divided by \( m \). For example, \( r = 86 \) is a quadratic residue of \( m = 7 \) because if we let \( x = 4 \) then \( x^2 = 16 \), and 16 and 86 each have a remainder of 2 when divided by 7. This is also written as there exists an integer \( x \) such that \( x^2 \equiv r \pmod{m} \). Find the sum of the smallest three positive prime numbers that are quadratic residues of \( m = 5 \).

**Answer:** 35
If \( r \) and \( m \) are positive integers, \( r \) is a quadratic residue of \( m \) if there exists an integer \( x \) such that the remainder when \( x^2 \) is divided by \( m \) is equal to the remainder when \( r \) is divided by \( m \). For example, \( r = 86 \) is a quadratic residue of \( m = 7 \) because if we let \( x = 4 \) then \( x^2 = 16 \), and 16 and 86 each have a remainder of 2 when divided by 7. This is also written as there exists an integer \( x \) such that \( x^2 \equiv r \pmod{m} \). Find the sum of the smallest three positive prime numbers that are quadratic residues of \( m = 5 \).

**Answer:**

35
Solution:

Squaring the integers from 0 to 10 and calculating the remainder when dividing by 5 yields the pattern of 0, 1, 4, 4, 1, 0, 1, 4, 4, 1, 0. This pattern repeats itself infinitely, so a prime number must have a remainder of 0, 1, or 4 to be a quadratic residue of 5. Prime numbers and their remainders when dividing by 5 are in the following table:

<table>
<thead>
<tr>
<th>Prime</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>13</th>
<th>17</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>Remainder</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

The first three primes with remainders of 0, 1, or 4 when dividing by 5 are 5, 11, and 19:

\[ 5 + 11 + 19 = 35 \]
Stop!

Do not continue
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to do so!
Round #7
(15 minutes)
Problem A (4 points)

Four numbers are written in a row. The mean of the first two numbers is 5. The mean of the middle two numbers is 4, and the mean of the last two numbers is 10. What is the mean of the first and last numbers?

Problem B (6 points)

In the figure, the length of sides of the larger square is 7. The length of sides of the smaller square is \( b \). Find the area of the triangle \( ABC \).

Problem C (10 points)

A triangular number is any number that can be expressed as \( \frac{1}{2}n(n + 1) \), where \( n \) is a nonnegative integer. For example, 21 is a triangular number because \( 21 = \frac{(6)(7)}{2} \). Carl Friedrich Gauss discovered that every positive integer can be expressed as a sum of exactly three (not necessarily distinct) triangular numbers, recording the famous words in his diary “EYPHKA! num = △ + △ + △”. Find three triangular numbers such that their sum is 2014, and such that the largest of the three numbers is as large as possible.
Answer: Problem A (4 points)

Answer: Problem B (6 points)

Answer: Problem C (10 points)
Round #7 (15 minutes)

Answer: Problem A (4 points)
11

Answer: Problem B (6 points)
\[ \frac{49}{2} = 24.5 \]

Answer: Problem C (10 points)
1953, 55, 6
Stop!

Do not continue until directed to do so!
Round #8
(10 points/10 minutes)
How many positive integer solutions does the equation $x + y + z + w = 15$ have?
How many positive integer solutions does the equation \( x + y + z + w = 15 \) have?

**Answer:** 364
How many positive integer solutions does the equation $x + y + z + w = 15$ have?

Answer: 364
Solution:

Imagine 15 red balls lined up in a row. Next, use three purple balls as dividers between the red balls. There are 14 different spots to place the three purple balls as dividers. Thus, the number of positive integer solutions to \( x + y + z + w = 15 \) is

\[
\binom{14}{3} = \frac{14 \cdot 13 \cdot 12}{3 \cdot 2 \cdot 1} = 7 \cdot 13 \cdot 4 = 364
\]
Stop!

Do not continue until directed to do so!
Tie-Breaker #1

(1 point)
The traffic light in front of the Laie Shopping Center has a 120-second cycle. The light is green for 80 seconds, yellow for 5 seconds and red for 35 seconds. Driving to and from the Math Bowl, what is the probability that the light will be red both times?
The traffic light in front of the Laie Shopping Center has a 120-second cycle. The light is green for 80 seconds, yellow for 5 seconds and red for 35 seconds. Driving to and from the Math Bowl, what is the probability that the light will be red both times?

Answer:
The traffic light in front of the Laie Shopping Center has a 120-second cycle. The light is green for 80 seconds, yellow for 5 seconds and red for 35 seconds. Driving to and from the Math Bowl, what is the probability that the light will be red both times?

Answer:

\[ \frac{49}{576} \]
Solution:

The probability that it is red when you drive by (assuming you do drive by it) is

\[
\frac{35}{120} = \frac{7}{24}.
\]

It follows that the probability that it is red both times is

\[
\left(\frac{7}{24}\right)^2 = \frac{49}{576}.
\]
Stop!

Do not continue until directed to do so!
Tie-Breaker #2

(1 point)
Two pulleys, one with radius 3 inches and the other with radius 10 inches, are connected by a belt. If the 3-inch pulley is caused to rotate at 3 revolutions per minute, at how many revolutions per minute does the 10-inch pulley rotate?
Two pulleys, one with radius 3 inches and the other with radius 10 inches, are connected by a belt. If the 3-inch pulley is caused to rotate at 3 revolutions per minute, at how many revolutions per minute does the 10-inch pulley rotate?

**Answer:**

[Blank space]
Two pulleys, one with radius 3 inches and the other with radius 10 inches, are connected by a belt. If the 3-inch pulley is caused to rotate at 3 revolutions per minute, at how many revolutions per minute does the 10-inch pulley rotate?

Answer:

\[
\frac{9}{10}
\]
Solution:

The distance around the smaller pulley is

\[
\text{distance traveled} = \text{circumference} \cdot \text{rev per minute}
\]

\[
= 6\pi \cdot 3 = 18\pi \text{ per minute}
\]

The circumference for the larger pulley is \(20\pi\), thus the ten inch pulley will travel

\[
\frac{18\pi}{20\pi} = \frac{9}{10} \text{ rev per minute}
\]
Stop!
Do not continue until directed to do so!
Tie-Breaker #3

(1 point)
Find the value of $x \sin x$ if $x = \frac{\pi}{6}$
Find the value of \( x \sin x \) if \( x = \frac{\pi}{6} \)

**Answer:**

\[
\frac{\pi}{6} \sin \left( \frac{\pi}{6} \right) = \frac{\pi}{6} \cdot \frac{1}{2} = \frac{\pi}{12}
\]
Find the value of $x \sin x$ if $x = \frac{\pi}{6}$

**Answer:**

\[
\frac{\pi}{12}
\]
Tie-Breaker #3
(1 point)

Find the value of \( x \sin x \) if \( x = \frac{\pi}{6} \)

Answer:

\[
\frac{\pi}{12}
\]

Solution:

\[
\frac{\pi}{6} \sin \frac{\pi}{6} = \frac{\pi}{6} \left( \frac{1}{2} \right) = \frac{\pi}{12}
\]