The Body Mass Index (BMI) varies directly with a person’s weight (pounds) and inversely with the square of the person’s height (inches). Given that a 6-foot tall man weighing 180 pounds has a BMI of 24.0, what is the BMI of a woman weighing 120 pounds with a height of 5 feet 4 inches?

Answer:

20.25

Solution:
The formula to work with is $BMI = \frac{kw}{h^2}$. It follows that

$$24 = \frac{180k}{72^2} \implies k = 691.2$$

So the BMI for the 5 foot 4 inch woman who weighs 120 pounds is

$$BMI = \frac{(691.2)(120)}{64^2} = 20.25$$
All answers must be expressed exactly

Find the remaining real zero for the 4th degree polynomial function with the following properties: it only has real coefficients, 1/2 and 1 + 2i are two of its zeros, its y-intercept is −30, and its leading coefficient is 2.

Answer: 

−6

Solution:

Because this is a polynomial with real coefficients, complex roots occur in conjugate pairs. Thus, three of the roots are 1/2, 1 + 2i, and 1 − 2i. Let the fourth root be c. The leading coefficient is 2 so the polynomial needs to equal 

\[2(x - 1/2)(x - (1 + 2i))(x - (1 - 2i))(x - c).\]

The constant term of this expression equals 

\[2(-1/2)(-1 - 2i)(-1 + 2i)(-c) = 5c,\]

but this is the y-intercept of the polynomial, so 5c = −30, and c = −6.
What is the remainder when $x^{2014} + x^{2013}$ is divided by $x^2 + 1$?

**Answer:**

$x - 1$

**Solution:** (Quick solution)

Dividing any polynomial by $x^2 + 1$ and taking the remainder is actually equivalent to substituting $x = i$, using $i^2 = -1$ to reduce the expression to $a + bi$, and then substituting $x$’s back in for $i$’s.

$$x^{2014} + x^{2013} \rightarrow i^{2014} + i^{2013} = -1 + i \rightarrow -1 + x = x - 1$$

So the answer is $x - 1$.

**Solution:** (Brute Force solution)

Performing four iterations of long division gives the following:

$$
\begin{array}{c}
   \underline{x^2 + 1} \\
   x^{2014} + x^{2013} \\
   \underline{-x^{2014} - x^{2012}} \\
   x^{2013} - x^{2012} \\
   \underline{-x^{2013} - x^{2011}} \\
   x^{2012} - x^{2011} \\
   \underline{-x^{2012} - x^{2010}} \\
   x^{2011} + x^{2010} \\
   \underline{-x^{2011} + x^{2009}} \\
   x^{2010} + x^{2009}
\end{array}
$$

At this point, the signs repeat themselves in sets of four. Thus, the division will continue until we get to $x^2 + x$. The last division will be

$$
\begin{array}{c}
   \underline{x^2 + 1} \\
   x^2 + x \\
   \underline{-x^2 - 1} \\
   x - 1
\end{array}
$$

Thus the remainder is $x - 1$.
Using the following figure (not drawn to scale), determine the value of $m\angle A + m\angle B + m\angle C + m\angle D$.

Answer:

$280^\circ$

Solution:
It can be seen directly that

$A + D + 125^\circ = 180^\circ \implies A + D = 55^\circ$.

Next, $B + C + E + F = 360^\circ$ and $E + F + 45^\circ = 180^\circ$.
Thus, $B + C = 225^\circ$.
It follows that

$A + B + C + D = 225^\circ + 55^\circ = 280^\circ$.
The sum of the lengths of the twelve edges of a closed rectangular box is 140, and the distance from one corner of the box to the farthest corner is 21. The total surface area of the box is

Answer:

784

Solution:

Let $h, w$, and $l$ be the height, width, and length of the box. Because the sum of the lengths of all twelve edges is 140, then $4h + 4w + 4l = 140$, and therefore $h + w + l = 35$. Because the distance from one corner to the most distant one is 21, then $h^2 + w^2 + l^2 = 21^2$. The surface area, $S$, of the box equals $2lh + 2lw + 2hw$. Now,

$$(h + w + l)^2 = h^2 + w^2 + l^2 + \frac{2lh + 2lw + 2hw}{S}$$

$35^2 = 21^2 + S$

$784 = S$
If $3 \sin \theta + 4 \cos \theta = 5$, then $\tan \theta =$?

**Answer:** \[
\frac{3}{4}
\]

**Solution:**

\[
3 \sin \theta + 4 \cos \theta = 5
\]
\[
4 \cos \theta = 5 - 3 \sin \theta
\]
\[
16 \cos^2 \theta = 25 - 30 \sin \theta + 9 \sin^2 \theta
\]
\[
16(1 - \sin^2 \theta) = 25 - 30 \sin \theta + 9 \sin^2 \theta
\]
\[
0 = 25 \sin^2 \theta - 30 \sin \theta + 9
\]

Using the quadratic formula gives

\[
\sin \theta = \frac{30 \pm \sqrt{30^2 - 4 \cdot 9 \cdot 25}}{2(25)} = \frac{30 \pm 3}{50} = \frac{3}{5}
\]

It must be that $\cos \theta > 0$ (otherwise the sum of 5 is impossible). This gives

\[
\cos \theta = \sqrt{1 - \left( \frac{3}{5} \right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}
\]

It follows that $\tan \theta = \frac{3}{4}$
In the figure, $ABCD$ is a square with side length of $a$. $E$ is the midpoint of $BC$. $AE$ intersects the diagonal $BD$ at $F$. Find the area of triangle $ABF$.

Answer:

$$\frac{a^2}{6}$$

Solution:
The slope of the line passing through $AE$ is $\frac{-a}{\frac{1}{2}a} = -2$. Thus, the equation of the line is $y = -2x + a$. The line passing through $BF$ is $y = x$. Their intersection point is

$$x = -2x + a \Rightarrow 3x = a \Rightarrow x = \frac{a}{3}$$

It follows that the height of the triangle $ABF$ is $\frac{a}{3}$ and the base is $a$. Thus, the area is

$$A = \frac{1}{2}bh = \frac{a^2}{6}$$
Find the sum of all \( x \)-values satisfying the equation \( 2 \cos 3x + 1 = 0 \) on the interval \([0, \pi]\).

**Answer:** \[ \frac{14\pi}{9} \]

**Solution:**

The solutions to the equation also solve \( \cos 3x = -\frac{1}{2} \). It follows that

\[
\begin{align*}
3x &= \frac{2\pi}{3} + 2\pi n \\
x &= \frac{2\pi}{9} + \frac{2\pi}{3} n
\end{align*}
\]

or

\[
\begin{align*}
3x &= -\frac{2\pi}{3} + 2\pi n \\
x &= -\frac{2\pi}{9} + \frac{2\pi}{3} n
\end{align*}
\]

Of those values above, the sum of those that result with angles between 0 and \( \pi \) are

\[
\frac{2\pi}{9} + \frac{4\pi}{9} + \frac{8\pi}{9} = \frac{14\pi}{9}
\]

Urn A contains 4 white balls and 6 black balls. Urn B contains 3 white balls and 5 black balls. A ball (the transferred ball) is drawn from urn A and then transferred to urn B. A ball (the second ball) is then drawn from urn B. What is the probability that the transferred ball was black given that the second ball drawn was white?

**Answer:** \( \frac{9}{17} \)

**Solution:**

Let \( T_B \) and \( T_W \) be the events that the transferred ball is black or white respectively. Further, let \( S \) be the event that the second ball drawn is white. It follows that

\[
P(T_B|S) = \frac{P(T_B \text{ and } S)}{P(S)} = \frac{P(T_B)P(S|T_B)}{P(T_B)P(S|T_B) + P(T_W)P(S|T_W)}
\]

\[
= \frac{\frac{6}{10} \cdot \frac{3}{9}}{\frac{6}{10} \cdot \frac{3}{9} + \frac{4}{10} \cdot \frac{4}{9}} = \frac{\frac{1}{5} \cdot \frac{8}{45}}{\frac{1}{5} + \frac{8}{45}}
\]

\[
= \frac{9}{17}
\]
If $r$ and $m$ are positive integers, $r$ is a quadratic residue of $m$ if there exists an integer $x$ such that the remainder when $x^2$ is divided by $m$ is equal to the remainder when $r$ is divided by $m$. For example, $r = 86$ is a quadratic residue of $m = 7$ because if we let $x = 4$ then $x^2 = 16$, and 16 and 86 each have a remainder of 2 when divided by 7. This is also written as there exists an integer $x$ such that $x^2 \equiv r \pmod{m}$. Find the sum of the smallest three positive prime numbers that are quadratic residues of $m = 5$.

**Answer:**

35

**Solution:**

Squaring the integers from 0 to 10 and calculating the remainder when dividing by 5 yields the pattern of

$$0, 1, 4, 4, 1, 0, 1, 4, 4, 1, 0.$$ 

This pattern repeats itself infinitely, so a prime number must have a remainder of 0, 1, or 4 to be a quadratic residue of 5. Prime numbers and their remainders when dividing by 5 are in the following table:

<table>
<thead>
<tr>
<th>Prime</th>
<th>Remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>4</td>
</tr>
</tbody>
</table>

The first three primes with remainders of 0, 1, or 4 when dividing by 5 are 5, 11, and 19:

$$5 + 11 + 19 = 35$$
Four numbers are written in a row. The mean of the first two numbers is 5. The mean of the middle two numbers is 4, and the mean of the last two numbers is 10. What is the mean of the first and last numbers?

Answer:

11

Solution:

It follows that

\[
\frac{x_1 + x_2}{2} = 5 \quad \frac{x_2 + x_3}{2} = 4 \quad \frac{x_3 + x_4}{2} = 10
\]

\[
x_1 + x_2 = 10 \quad x_2 + x_3 = 8 \quad x_3 + x_4 = 20
\]

Adding the first and last equations and subtracting the second yields:

\[
\begin{align*}
x_1 + x_2 &= 10 \\
-x_2 - x_3 &= -8 \\
x_3 + x_4 &= 20 \\
x_1 + x_4 &= 22
\end{align*}
\]

Thus, it follows that the mean of the first and last numbers are

\[
\frac{x_1 + x_4}{2} = 11.
\]
In the figure, the length of sides of the larger square is 7. The length of sides of the smaller square is $b$. Find the area of the triangle $ABC$.

Answer:

$$\frac{49}{2} = 24.5$$

Solution: (1)
The area of $\triangle ABC$ equals the area of the trapezoid $ABED$ plus the area of the triangle $BCE$ minus the area of $\triangle ACD$, which equals

$$\text{Area of } \triangle ABC = \text{Area of } ABED + \text{Area of } \triangle BCE - \text{Area of } \triangle ACD$$

$$= \frac{1}{2}b(b + 7) + \frac{49}{2} - \frac{1}{2}b(b + 7)$$

$$= \frac{49}{2} = 24.5$$

Solution: (2)
Extend the line segment $CB$ until it intersects with the vertical line passing through $A$, as seen in the figure. It follows that $AF = 7$ because angle $DCB$ is $45^\circ$. It follows that

$$\text{Area of } \triangle ABC = \text{Area of } \triangle FDC - \text{Area of } \triangle ADC - \text{Area of } \triangle ABF$$

$$= \frac{1}{2}(7 + b)^2 - \frac{1}{2}(7 + b)b - \frac{1}{2}(7)(b)$$

$$= \frac{1}{2} [49 + 14b + b^2 - 7b - b^2 - 7b]$$

$$= \frac{49}{2} = 24.5$$
A triangular number is any number that can be expressed as $\frac{1}{2}n(n + 1)$, where $n$ is a nonnegative integer. For example, 21 is a triangular number because $21 = \frac{(6)(7)}{2}$. Carl Friedrich Gauss discovered that every positive integer can be expressed as a sum of exactly three (not necessarily distinct) triangular numbers, recording the famous words in his diary “EYPHKA! $\text{num} = \triangle + \triangle + \triangle$”. Find three triangular numbers such that their sum is 2014, and such that the largest of the three numbers is as large as possible.

Answer:

1953, 55, 6

Solution:

The largest triangular number that is less than or equal to 2014 is

$$1953 = \frac{(62)(63)}{2}$$

The first 11 triangular numbers are 0, 1, 3, 6, 10, 15, 21, 28, 26, 45, and 55.

$$2014 = 1953 + 55 + 6$$

No other combination involving 1953 works. So the numbers are 1953, 55, and 6.
How many positive integer solutions does the equation $x + y + z + w = 15$ have?

Answer: 364

Solution:
Imagine 15 red balls lined up in a row. Next, use three purple balls as dividers between the red balls. There are 14 different spots to place the three purple balls as dividers. Thus, the number of positive integer solutions to $x + y + z + w = 15$ is

$$\binom{14}{3} = \frac{14 \cdot 13 \cdot 12}{3 \cdot 2 \cdot 1} = 7 \cdot 13 \cdot 4 = 364$$
The traffic light in front of the Laie Shopping Center has a 120-second cycle. The light is green for 80 seconds, yellow for 5 seconds and red for 35 seconds. Driving to and from the Math Bowl, what is the probability that the light will be red both times?

Answer: \[
\frac{49}{576}
\]

Solution:

The probability that it is red when you drive by (assuming you do drive by it) is

\[
\frac{35}{120} = \frac{7}{24}.
\]

It follows that the probability that it is red both times is

\[
\left(\frac{7}{24}\right)^2 = \frac{49}{576}.
\]
Two pulleys, one with radius 3 inches and the other with radius 10 inches, are connected by a belt. If the 3-inch pulley is caused to rotate at 3 revolutions per minute, at how many revolutions per minute does the 10-inch pulley rotate?

Answer: \[
\frac{9}{10}
\]

Solution:

The distance around the smaller pulley is

\[
\text{distance traveled} = \text{circumference} \cdot \text{rev per minute}
\]

\[= 6\pi \cdot 3 = 18\pi \text{ per minute}\]

The circumference for the larger pulley is \(20\pi\), thus the ten inch pulley will travel

\[
\frac{18\pi}{20\pi} = \frac{9}{10} \text{ rev per minute}
\]
Find the value of $x \sin x$ if $x = \frac{\pi}{6}$

Answer: \[
\frac{\pi}{12}
\]

Solution:

\[
\frac{\pi}{6} \sin \frac{\pi}{6} = \frac{\pi}{6} \left( \frac{1}{2} \right) = \frac{\pi}{12}
\]