Kimo paddles upstream for 8 hours. If his rate of paddling is twice the rate of the current, at what time will he have to start heading downstream in order to reach his starting point at 6pm?

Answer:

3:20 pm

Solution:

Let $r_1$ be the rate of the paddling and $r_2$ be the rate of the current. Then it follow that $r_1 = 2r_2$. The distance up the stream is the same as down the stream. Thus

\[ (r_1 + r_2)t = 8(r_1 - r_2) \]

\[ t = \frac{8(r_1 - r_2)}{r_1 + r_2} = \frac{8(2r_2 - r_2)}{2r_2 + r_2} = \frac{8}{3} \]

It will take him $8/3$ hrs to get back down, which is 2 hours and 40 minutes. He should therefore start at 3:20pm.
A math professor, basketball coach, and their student went to Burger Queen for lunch. The math professor paid $14.25 for 8 hamburgers, 5 orders of fries, and 2 cokes. The basketball coach paid $8.51 for 5 hamburgers, 3 orders of fries, and 1 coke. What did the student pay for 1 hamburger, 1 order of fries, and 1 coke?

Answer:

$2.97

Solution:

The professor and coach’s order leads to the system of equations

\[
\begin{align*}
14.25 &= 8h + 5f + 2c \\
8.51 &= 5h + 3f + c
\end{align*}
\]

Multiplying the math professor’s equation by 2 and multiplying the basketball coach’s equation by 3 and subtracting leads to

\[
\begin{align*}
28.50 &= 16h + 10f + 4c \\
25.53 &= 15h + 9f + 3c \\
2.97 &= h + f + c
\end{align*}
\]

Thus, it follows that the student paid $2.97
Find all real solutions for $x$:

$$(x^2 + 2x - 24)^{x^3 - 9x^2 + 20x} = 1$$

Answer:

$0, 5, -1 \pm \sqrt{26}$

Solution:

The equation will equal 1 where the exponent is zero (and the base isn’t zero), and where the base is equal to one. First, the exponent is zero when

$$0 = x^3 - 9x^2 + 20x = x(x^2 - 9x + 20) = x(x - 5)(x - 4)$$

It follows that $x = 0$, $x = 5$, or $x = 4$. However, the base $x^2 + 2x - 24 = 0$, when $x = 4$, so we must toss this solution out. Next, we find were

$$x^2 - 2x - 24 = 1 \implies x^2 - 2x - 25 = 0.$$ 

It follows that

$$x = \frac{-2 \pm \sqrt{4 + 4(25)}}{2} = \frac{-2 \pm 2\sqrt{26}}{2} = -1 \pm \sqrt{26}$$
A circle has both inscribed and circumscribed squares. The difference of their areas is 40 square units. What is the circumference of the circle?

Answer:

$$4\sqrt{5}\pi$$

Solution:

As seen in the diagram, the area of the circumscribed square is $(2r)^2 = 4r^2$. The inscribed square consists of four right triangles with legs of length $r$. Thus, the area of the inscribed square is $4\left(\frac{1}{2}r^2\right) = 2r^2$. The difference between the areas is 40.

We can solve for $r$ like so:

$$4r^2 - 2r^2 = 40 \implies 2r^2 = 40 \implies r^2 = 20 \implies r = \sqrt{20} = 2\sqrt{5}$$

It follows that the circumference is

$$C = 2\pi r = 2\pi(2\sqrt{5}) = 4\sqrt{5}\pi$$
Chris and Pat work at Hawaiian Electric. One day, Chris said to Pat, “19/40 of my coworkers are female”. Pat replied “12/25 of my coworkers are female”. If Chris and Pat are of different genders and they are both correct, then how many workers are there at Hawaiian Electric?

Answer:

201 workers

Solution:
Pat and Chris have the same number of co-workers. However, they differ in female co-workers by 1. Since $\frac{19}{40} = .475 < \frac{12}{25} = .48$, then Chris must be female. Suppose there are $w$ co-workers. Then it follows that

$$\frac{f}{w} = \frac{19}{40} \implies f = \frac{19}{40}w$$

and

$$\frac{f + 1}{w} = \frac{12}{25} \implies 25(f + 1) = 12w$$

$$25\left(\frac{19}{40}w + 1\right) = 12w$$

$$\frac{95}{8}w + 25 = 12w$$

$$-\frac{1}{8}w = -25$$

$$w = 200$$

Because either Chris or Pat would have 200 co-workers, there is a total of $w + 1 = 201$ workers.
An army of ants is marching across the kitchen floor. If they form columns with 10 ants in each column, then there are 6 ants left over. If they form columns of 7, 11, or 13 ants in each column, then there would be 2 ants left over. What is the smallest number of ants that could be in this army?

Answer:

4006 ants

Solution: (Logic)

Let $t$ be the total number of ants. Because groups of 7, 11, or 13 produce two ants left over, $t - 2$ is a multiple of 7, 11, and 13. Hence $t - 2$ is a multiple of the least common multiple of 7, 11, and 13; because those numbers are relatively prime, $t - 2$ is therefore a multiple of $7 \times 11 \times 13 = 1001$. We are also given that groups of 10 ants produce 6 ants left over, so the last digit of $t$ is 6. Given that $t$ is 2 more than a multiple of 1001, the first $t$ to satisfy all these conditions is 4006.

Solution: (Using modular arithmetic)

Suppose $t$ is the total number of ants. According to the description, it follows that

$$t = 6(\text{mod } 10) = 2(\text{mod } 7) = 2(\text{mod } 11) = 2(\text{mod } 13)$$

It follows that

$$t = 7w + 2 = 2(\text{mod } 11)$$

$$7w = 0(\text{mod } 11)$$

$$w = 0(\text{mod } 11) = 11d$$

This leads to

$$t = 7(11d) + 2 = 2(\text{mod } 13)$$

$$77d = 0(\text{mod } 13)$$

$$d = 0(\text{mod } 13) = 13e$$

Lastly,

$$t = 77(13e) + 2 = 6(\text{mod } 10)$$

$$1001e = 4(\text{mod } 10)$$

$$e = 4(\text{mod } 10) = 10g + 4$$

Therefore, the possible number of ants is

$$t = 1001(10g + 4) + 2 = 10010g + 4006$$

If $g = 0$, we have the least possible number of ants, 4006.
A can of macadamia nuts has a 3 inch diameter and 4 inch height. What is the maximum number of cans that will fit in a box that measures 24 inches long, 24 inches wide, and 4 inches high on the inside?

Answer:

68

Solution:

Since the cans are circular, then the smallest repeatable configuration is as drawn on the right. Note that if we place the cans in the box on their sides, then it is only possible to place one layer of cans (since two layers would be too tall). In this case we would have \( 8 = 48 \) cans. Suppose that they are standing up. It follows that the number of rows of triangles possible would be

\[
\frac{3}{2} + \frac{3\sqrt{3}}{2} t + \frac{3}{2} < 24 \implies \frac{3\sqrt{3}}{2} t < 21 \implies r < \frac{14\sqrt{3}}{3} \approx 8.08
\]

Thus, it is possible to have 9 rows of cans. If we start with a row of 8, then alternate with rows of 7 and 8, then we will have 5 rows of 8 and 4 rows of 7. Thus there are

\[ 8 \cdot 5 + 7 \cdot 4 = 40 + 28 = 68 \]

cans.
Triangle $ABC$ is an isosceles right triangle with legs $AC = BC = 1$. Triangle $ADC$ is an equilateral triangle with sides of length 1. If $E$ is the point of intersection of $AB$ and $CD$, what is the area of triangle $ADE$?

**Answer:**

$$\frac{\sqrt{3}}{2} - \frac{3}{4} \text{ or } \frac{2\sqrt{3} - 3}{4}$$

**Solution:**

We need to find the coordinates of the point $E$. Since $ADC$ is an equilateral triangle, then it follows that $\triangle FDC$ is a $30^\circ - 60^\circ - 90^\circ$ triangle, thus $CF = \frac{1}{2}$ and $DF = \frac{\sqrt{3}}{2}$. It follows that the equation of the line $DC$ is $y = \sqrt{3}x$. Similarly, the line $AB$ is $y = -x + 1$. The intersection of the two lines occurs when

$$\sqrt{3}x = -x + 1 \implies x = \frac{1}{\sqrt{3} + 1}$$

It follows that the distance $EG$ is the $y$ value:

$$y = \sqrt{3}x = \sqrt{3} \left( \frac{1}{\sqrt{3} + 1} \right) = \frac{\sqrt{3}}{\sqrt{3} + 1} \left( \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \right) = \frac{3 - \sqrt{3}}{2}$$

Thus, the area of $\triangle ADE$ is:

$$\text{Area}_{\triangle ADE} = \text{Area}_{\triangle ADC} - \text{Area}_{\triangle AEC}$$

$$= \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) (1) - \frac{1}{2} \left( \frac{3 - \sqrt{3}}{2} \right) (1)$$

$$= \frac{\sqrt{3}}{4} - \frac{3}{4} + \frac{\sqrt{3}}{4}$$

$$= \frac{\sqrt{3}}{2} - \frac{3}{4} = \frac{2\sqrt{3} - 3}{4}$$
The mean score for those who passed the last test in Dr. Furuto’s class was 65, while the mean score for those who failed the test was 35. If the mean for the entire class was 53, what percentage of the class passed the test?

Answer:

60%

Solution:

Let \( p \) be the number in the class that passed and \( f \) be the number that failed. Let \( \bar{x}_p = 65 \) be the average of those who passed, \( \bar{x}_f = 35 \) be the average of those who failed, and \( \bar{x} = 53 \) be the average of the class. Suppose we order the students in the class with the first \( p \) are those that passed, and the last \( f \) are those that failed. It follows that

\[
\bar{x} = 53 = \frac{x_1 + x_2 + \cdots + x_p + x_{p+1} + \cdots + x_{f+p}}{p + f}
\]

\[
= \frac{x_1 + x_2 + \cdots + x_p}{p} + \frac{x_{p+1} + \cdots + x_{f+p}}{p + f}
\]

\[
= \frac{65p + 35f}{p + f}
\]

\[
53(p + f) = 65p + 35f
\]

\[
18f = 12p
\]

\[
f = \frac{2}{3}p
\]

It follows that the percentage of the class that passed is therefore

\[
\frac{p}{p + f} = \frac{p}{p + \frac{2}{3}p} = \frac{1}{1 + \frac{2}{3}} = \frac{3}{5}
\]

Hence, 60% of the class passed the test.
Find a four digit integer, \( x \), such that \( 4x \) is another four-digit number whose digits are in the reverse order of the digits of \( x \).

**Answer:**

2178

**Solution:**

(1) Suppose the four digit number is \( ABCD \). Then

\[
4(1000A + 100B + 10C + D) = 1000D + 100C + 10B + A \\
1333A + 130B = 20C + 332D
\]

If \( A \) is 3 or larger, then both \( C \) and \( D \) could not be single digits. Also, the coefficients of \( B \), \( C \), and \( D \) are even, so \( A \) cannot be odd. Therefore \( A \) is 2. Substituting and simplifying further yields

\[
1333 + 65B = 10C + 166D
\]

Because the coefficients of \( C \) and \( D \) are both even, then the sum on the left must be even; thus \( B \) must be odd. When trying different possible values of \( B \), \( D \) can be determined quickly because \( C \) cannot affect the ones’ digit of the sum on the right. Of the possible combinations only \( B = 1 \) works, where \( C = 7 \), and \( D = 8 \).

**Solution:** (alternative)

Suppose that the 4 digit number is \( abcd \). Since \( 4a \) is also a 4 digit number, then it follows that \( 4a < 10 \). So \( a = 1, 2 \). Next, since \( 4d = a(\text{mod } 10) \), then \( a \) cannot be 1. Thus, \( a = 2 \), and \( d = 4a = 8 \). Similarly, since \( a = 2 \), then it follows that \( 4b < 10 \). Thus, \( b = 1, 2 \). Finally, the problem simplifies to

\[
\begin{array}{cccc}
2 & 1 & c & 8 \\
\times & 4 & & \\
8 & c & 1 & 2 \\
\end{array}
\quad \text{or} \quad
\begin{array}{cccc}
2 & 2 & c & 8 \\
\times & 4 & & \\
8 & c & 2 & 2 \\
\end{array}
\]

If \( b = 2 \), then \( 4c + 3 = 2(\text{mod } 10) \), which is impossible. It must be that \( b = 1 \). Thus,

\[
4c + 3 = 1(\text{mod } 10) \implies 4c = -2(\text{mod } 10) \implies 4c = 8(\text{mod } 10)
\]

Thus, \( c = 2 \) or \( c = 7 \). The solution \( c = 7 \) satisfies our constraints and

\[
\begin{array}{cccc}
2 & 1 & 7 & 8 \\
\times & 4 & & \\
8 & 7 & 1 & 2 \\
\end{array}
\]

Therefore, the 4 digit number is 2178.
Suppose that $0 \leq x \leq 2\pi$. Find all solutions to the equation

$$\frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\cos x - \sin x} = \frac{\tan 2x + \cot 2x}{\sec 2x}$$

Answer:

$$\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

Solution:

$$\frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\cos x - \sin x} = \frac{\tan 2x + \cot 2x}{\sec 2x}$$

$$\frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\cos x - \sin x} = \frac{\sin 2x + \cos 2x}{\cos 2x + \sin 2x} \cos 2x$$

$$\frac{\cos 2x}{\cos 2x + \sin 2x} = \frac{1}{\sin 2x}$$

The solutions are where

$$2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

Thus,

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$
How many integers with 4 different digits are there between 1,000 and 9,999 such that the absolute value of the difference between the first digit and the last digit is 2?

Answer: 840

Solution:

There are nine choices for the first digit of any of these numbers. The last digit sometimes has one choice, but other times there are two choices. Irregardless, the middle two digits can be chosen $8 \cdot 7 = 56$ ways. In particular, the digits that are possible are given in in the following table:

\[
\begin{array}{ccc}
1 & \_ & \_ & 3 \\
2 & \_ & \_ & 0,4 \\
3 & \_ & \_ & 1,5 \\
4 & \_ & \_ & 2,6 \\
5 & \_ & \_ & 3,7 \\
6 & \_ & \_ & 5,9 \\
7 & \_ & \_ & 6 \\
8 & \_ & \_ & 7 \\
\end{array}
\]

For numbers starting with 1, 8, and 9, there are $8 \times 7$ possibilities, whereas for numbers 2–7, there are $2 \times 8 \times 7$ possibilities. Thus, the number of possibilities is

$$3(8 \times 7) + 6(2 \times 8 \times 7) = 15(8 \times 7) = 840$$
Find the sum of the digits in the following product:

\[3,333,333,333 \times 3,333,333,333.\]

Answer:

90

Solution: (Algebraic)

\[3,333,333,333 \times 3,333,333,333 = 1,111,111,111 \times 9,999,999,999\]
\[= 1,111,111,111 \times (10,000,000,000 - 1)\]
\[= 11,111,111,110,000,000,000 - 1,111,111,111\]

This will be a number starting with nine ones, a zero, and then nine 8’s followed by a 9. The sum of the digits is therefore \(9 \times 1 + 9 \times 8 + 9 = 90\).

Solution: (Generalized)

Consider the more general problem \((3\cdots3)^2\). In the case of this problem, \(n = 10\). If we start with smaller values of \(n\), we can find a pattern in the sum of the digits. For example, \(n = 2\), the multiplication is 1089, which has the sum of \(1 \cdot 1 + 1 \cdot 8 + 9\). For \(n = 3\), the multiplication is 110889, which has the sum of \(2 \cdot 1 + 2 \cdot 8 + 9\). It follows the pattern that develops for \(n\) digits of threes is

\[(n - 1) \cdot 1 + (n - 1) \cdot 8 + 9 = 9n\]

For \(n = 10\) digits, the sum of the digits of the multiplication is \(9(10) = 90\).
A 3-4-5 right triangle is drawn so that the endpoints of the hypotenuse are (0,0) and (5,0). A square with area 1 is drawn so that its center is the vertex of the right triangle and its sides are parallel to the $x$ and $y$ axes. What is the area of the region where the triangle and square overlap?

Answer: \[
\frac{1}{4}
\]

Solution:
In the figure on the right, the rectangle corresponds to the bottom half of the square of area 1, and the point $B$ corresponds to the top of the 3-4-5 triangle as described. It follows that $\triangle ABF$ and $\triangle BCD$ are 3-4-5 triangles as well.

The area of the overlap is the area of $\triangle ABF$ plus the area of the trapezoid $FBDE$. However, since $\triangle ABF$ and $\triangle BCD$ are exactly the same (they are similar triangles and one side is the same), the area of the overlap is the same as the area of the square $BCEF$, which is $\frac{1}{4}$.

Solution: (2)
Within the square, extend the legs of the right triangle. The result is that there are two perpendicular lines intersecting in the center of the square. Because of symmetry, these two lines divide the square into four identical regions irrespective of the orientation of the triangle. The area of one of these regions, which is the area of overlap, is $\frac{1}{4}$. 
Find the tens digit in the sum

\[(1!)^3 + (2!)^3 + (3!)^3 + (4!)^3 + \cdots + (100!)^3\]

Answer:

4

Solution:

When 100 is a factor of a number, the number ends in two zeros and can add nothing to the tens digit of the sum. Since \((5!)^3\) has 100 as a factor (and so does every other term \((n!)^3\), where \(n > 5\)), it follows that the tens digit of the entire sum would have the same tens digit as the sum of the first four terms. Now,

\[(1!)^3 + (2!)^3 + (3!)^3 + (4!)^3 = 1 + 8 + 216 + 13824 = 14049\]

Thus, the tens digit is 4.
If a man died 1 billion seconds after he was born, what age in years was printed on his death certificate?

Answer: 31

Solution:

$$1 \times 10^9 \text{sec} \times \frac{1 \text{min}}{60 \text{sec}} \times \frac{1 \text{hr}}{60 \text{min}} \times \frac{1 \text{day}}{24 \text{hrs}} \times \frac{1 \text{years}}{365.25 \text{days}} = 31.7 \text{ years}$$

Thus, he is 31 years old when he dies.
The smallest of three numbers is $\log 1$. Their mean is $\log 4$. Their median is $\log 5$. Find the largest number.

**Answer:**

$$\log \left( \frac{64}{5} \right)$$

**Solution:**

Let $x < y < z$. It follows that $y = \log 5$ (it is the median). Since their average is $\log 4$, it follows that

$$\frac{\log 1 + \log 5 + z}{3} = \log 4$$

$$z + \log 5 = \log 4^3$$

$$z = \log \left( \frac{64}{5} \right)$$